Solution 8.5

The Hamiltonian $\hat{H}_0$ of the unperturbed system has solutions to the time-independent Schrödinger equation given by

$$\hat{H}_0|n\rangle = E_n|n\rangle$$

The time-independent eigenvalues are $E_n = \hbar\omega_n$ and the orthonormal eigenfunctions are $|n\rangle$. The eigenfunction $|n\rangle$ evolves in time according to

$$|n\rangle e^{-i\omega_n t} = \phi_n(x)e^{-i\omega_n t}$$

and satisfies

$$i\hbar\frac{\partial}{\partial t}|n\rangle e^{-i\omega_n t} = \hat{H}_0|n\rangle e^{-i\omega_n t}$$

At time $t = 0$ we apply a time-dependent change in potential $\hat{W}(x, t)$ whose effect is to create a new Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{W}(x, t)$$

and state $|\psi(x, t)\rangle$ which evolves in time according to

$$i\hbar\frac{\partial}{\partial t}|\psi(x, t)\rangle = (\hat{H}_0 + \hat{W}(x, t))|\psi(x, t)\rangle$$

The wave function $|\psi(x, t)\rangle$ may be expressed as a sum over the known unperturbed eigenstates

$$|\psi(x, t)\rangle = \sum_n a_n(t)|n\rangle e^{-i\omega_n t}$$

where $a_n(t)$ are time dependent coefficients. It follows that

$$i\hbar\frac{d}{dt}\sum_n a_n(t)|n\rangle e^{-i\omega_n t} = (\hat{H}_0 + \hat{W}(x, t))\sum_n a_n(t)|n\rangle e^{-i\omega_n t}$$

$$i\hbar\sum_n\left(\left(\frac{\partial}{\partial t}a_n(t)\right)|n\rangle e^{-i\omega_n t} + a_n(t)\left(\frac{\partial}{\partial t}|n\rangle e^{-i\omega_n t}\right)\right) = (\hat{H}_0 + \hat{W}(x, t))\sum_n a_n(t)|n\rangle e^{-i\omega_n t}$$

$$i\hbar\sum_n |n\rangle e^{-i\omega_n t}\frac{\partial}{\partial t}a_n(t) = \sum_n a_n(t)\,\hat{W}(x, t)|n\rangle e^{-i\omega_n t}$$

Multiplying both sides by $\langle m|$ and using the orthonormal relationship $\langle m|n\rangle = \delta_{mn}$ gives

$$i\hbar\frac{d}{dt}a_m(t) = \sum_n a_n(t)\langle m|\hat{W}(x,t)|n\rangle e^{i\omega_{mn}t}$$

If the perturbation is harmonic and of the form

$$\hat{W}(x,t) = V(x)\cos(\omega t)$$

where the spatial part of the potential is given by $V(x)$ then the term

$$\langle m|\hat{W}(x,t)|n\rangle e^{i\omega_{mn}t}$$

can be written as

$$\langle m|V(x)|n\rangle e^{i(\omega_m-\omega_n)t}\cos(\omega t) = \langle m|V(x)|n\rangle\frac{e^{i(\omega_{mn}+\omega)t}+e^{i(\omega_{mn}-\omega)t}}{2}$$

where $\omega_{mn} = \omega_m - \omega_n$ and the matrix elements in the representation of the unperturbed system described by Hamiltonian $\hat{H}_0$ are $\langle m|V(x)|n\rangle$.

Assuming an initial condition such that $a_n(t \le 0) = 1$ for only one eigenvalue and $a_m(t \le 0) = 0$ for $m \ne n$, then we may write

$$a_m(t) = \frac{1}{i\hbar}\int_{t'=0}^{t'=t}\langle m|V(x)|n\rangle\frac{e^{i(\omega_{mn}+\omega)t'}+e^{i(\omega_{mn}-\omega)t'}}{2}dt'$$

Performing the integration from time $t' = 0$ to $t' = t$ gives

$$a_m(t) = \frac{-\langle m|V(x)|n\rangle}{2\hbar}\left(\frac{e^{i(\omega_{mn}+\omega)t}-1}{\omega_{mn}+\omega}+\frac{e^{i(\omega_{mn}-\omega)t}-1}{\omega_{mn}-\omega}\right)$$

Taking the static limit $\omega \to 0$

$$a_m(t) = \frac{-\langle m|V(x)|n\rangle}{\hbar}\left(\frac{e^{i\omega_{mn}t}-1}{\omega_{mn}}\right)$$

$$a_m(t) = \frac{-\langle m|V(x)|n\rangle}{\hbar}e^{i\omega_{mn}t/2}\left(\frac{e^{i\omega_{mn}t/2}-e^{-i\omega_{mn}t/2}}{\omega_{mn}}\right)$$

$$a_m(t) = \frac{-2i\langle m|V(x)|n\rangle}{\hbar}e^{i\omega_{mn}t/2}\left(\frac{\sin(\omega_{mn}t/2)}{\omega_{mn}}\right)$$

where we used $2i\sin(x) = e^{ix}-e^{-ix}$. Assuming independent scattering channels, the probability of a transition out of state $|n\rangle$ into any state $|m\rangle$ is the *sum*

$$P_n(t) = \sum_m |a_m(t)|^2$$

If there are $D(E) = dN/dE$ states in the energy interval $dE = \hbar d\omega'$, then the sum can be written as an integral

$$P_n(t) = \frac{4}{\hbar^2}|\langle m|V(x)|n\rangle|^2\frac{dN}{dE}\int\frac{\sin^2((\omega'-\omega_n)t/2)}{(\omega'-\omega_n)^2}\hbar d\omega'$$

To perform the integral we change variables so that $x = (\omega'-\omega_n)/2$ and then take the limit $t \to \infty$. This gives

$$\left.\frac{\sin^2(tx)}{\pi t x^2}\right|_{t\to\infty} = \delta(x)$$

We now note that $dE/dx = 2\hbar$ since $E = \hbar 2x$. Hence, the integral can be written

$$\int \frac{\sin^2((\omega-\omega_n)t/2)}{(\omega-\omega_n)^2}\frac{dE}{dx}dx = 2\hbar\int\frac{\sin^2(tx)\ t\pi}{\pi t4x^2}dx = \frac{\hbar\pi t}{2}\int\frac{\sin^2(tx)}{\pi tx^2}dx$$

so that in the limit $t \to \infty$

$$\int \frac{\sin^2((\omega-\omega_n)t/2)}{(\omega-\omega_n)^2}\frac{dE}{dx}dx = \frac{\hbar\pi t}{2}\int\delta(x)dx$$

One may now write the probability of a transition out of state $|n\rangle$ into any state $|m\rangle$ as

$$P_n(t) = \frac{4}{\hbar^2}|\langle m|V(x)|n\rangle|^2 D(E)\frac{\hbar\pi t}{2} = \frac{2\pi}{\hbar}|W_{mn}|^2 D(E)t$$

Recognizing $dP_n(t)/dt$ as the inverse probability lifetime τ_n of the state $|n\rangle$, we can write Fermi's golden rule

$$\frac{1}{\tau_n} = \frac{2\pi}{\hbar}|\langle m|V(x)|n\rangle|^2 D(E)\delta(\omega_m-\omega_n)$$

The delta function is included to ensure energy conservation. Because we took the limit $\omega \to 0$ the final state energy must be the same as the initial state energy. If we did not take the limit $\omega \to 0$ so that $\omega \neq 0$ then the final state energy could be $\pm\hbar\omega$.

We have shown is that Fermi's golden rule can also be applied to the static limit of a harmonic perturbation. The significance of the result is that we can use time dependent perturbation theory to describe scattering from static potentials such as electrons scattering elastically from ionized impurities in a semiconductor.

Solution 8.6

(a) and (b) see text.

(c) Here it is important to recognize that there are different limiting cases to consider: (i) Perfectly ordered impurities in a periodic lattice, (ii) clusters of impurity scattering sites and (iii) anticlustering of scattering sites.

Solution 8.8

An electron mass m_0 and charge e is initially in the ground state of a one dimensiona harmonic oscillator characterized by frequency ω_0. At time $t = 0$ a uniform electri field $\mathbf{E}$ is applied in the x direction for time τ. This means that the time dependent per turbation potential is $\hat{W}(x, t) = -e|\mathbf{E}|\hat{x}$ for a time period τ. Applying first-order time dependent pertubation theory we have probability of excitation from the ground stat $|0\rangle$ to the first excited state $|1\rangle$

$$P_{01} = \frac{e^2|\mathbf{E}|^2}{\hbar^2}|\langle 1|\hat{x}|0\rangle|^2\left|\int_{t'=0}^{t'=\tau} e^{i\omega_0 t'}dt'\right|^2 = \frac{e^2|\mathbf{E}|^2}{\hbar^2}\left(\frac{\hbar}{2m_0\omega_0}\right)|\langle 1|(\hat{b}+\hat{b}^\dagger)|0\rangle|^2\left|\frac{e^{i\omega_0\tau}}{i\omega_0}-\frac{1}{i\omega_0}\right|^2$$

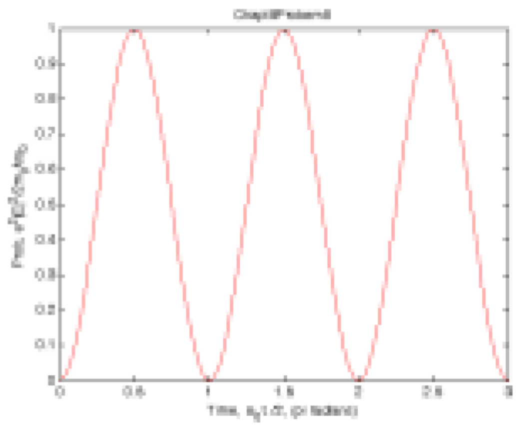
Noting that $|\langle 1|(\hat{b}+\hat{b}^\dagger)|0\rangle|^2 = 1$ and

$$\frac{e^{i\omega_0\tau}}{i\omega_0}+\frac{1}{i\omega_0} = \frac{e^{i\omega_0\tau/2}}{i\omega_0}(e^{i\omega_0\tau/2}-e^{-i\omega_0\tau/2}) = e^{i\omega_0\tau/2}\frac{\sin(\omega_0\tau/2)}{(\omega_0/2)}$$

we have

$$P_{01}(\tau) = \frac{e^2|\mathbf{E}|^2}{2m_0\hbar\omega_0}\frac{\sin^2(\omega_0\tau/2)}{(\omega_0/2)^2}$$

which has value zero when $\omega_0\tau/2 = n\pi$ where $n = 0, 1, 2, \ldots$.



Solution 8.9

(a) We start with the first-order time-dependent perturbation theory result for transition probability

$$P(t) = \frac{1}{\hbar^2}\sum_f\left|\int_{t'=0}^{t'=t}\langle\psi_f|\hat{W}(t')|\psi_i\rangle e^{i\omega_{fi}t'}dt'\right|^2$$

$$= \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}|\psi_i\rangle|^2\left|\int_{t'=0}^{t'=t}e^{i(\omega_{fi}+\omega)t'}dt'\right|^2 + \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2\left|\int_{t'=0}^{t'=t}e^{i(\omega_{fi}-\omega)t'}dt'\right|^2$$

which assumes that each scattering process is an independent parallel channel and where $\hbar\omega_{fi} = (E_f - E_i)$.

Performing the integration we have

$$P(t) = \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}|\psi_i\rangle|^2\left|\frac{e^{i(\omega_{fi}+\omega)t}-1}{(\omega_{fi}+\omega)}\right|^2 + \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2\left|\frac{e^{i(\omega_{fi}-\omega)t}-1}{(\omega_{fi}-\omega)}\right|^2$$

and using the relation $|e^x - 1|^2 = 4\sin^2(x/2)$ gives

$$P(t) = \frac{4W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}|\psi_i\rangle|^2\frac{\sin^2((\omega_{fi}+\omega)t/2)}{(\omega_{fi}+\omega)^2 t}t + \frac{4W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2\frac{\sin^2((\omega_{fi}-\omega)t/2)}{(\omega_{fi}-\omega)^2 t}t$$

(b) The equation in (a) can be re-written as

$$P(t) = \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}|\psi_i\rangle|^2\frac{\sin^2\left(\frac{(\omega_{fi}+\omega)}{2}t\right)}{\left(\frac{\omega_{fi}+\omega}{2}\right)^2 t}t + \frac{W_0^2}{\hbar^2}|\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2\frac{\sin^2((\omega_{fi}-\omega)t/2)}{\left(\frac{\omega_{fi}-\omega}{2}\right)^2 t}t$$

In the limit $t \rightarrow \infty$ we use $\delta(x) = \frac{1}{\pi} \lim_{\eta \rightarrow \infty} \frac{\sin^2(\eta x)}{\eta x^2}$. Setting $x = (\omega_{fi} + \omega)/2$ for the first term on the right hand side and $x = (\omega_{fi} - \omega)/2$ for the second term we have

$$P(t) = \frac{W_0^2}{\hbar^2} |\langle\psi_f|\hat{b}|\psi_i\rangle|^2 \left(\pi\delta\left(\frac{(\omega_{fi} + \omega)}{2}\right)\right) t + \frac{W_0^2}{\hbar^2} |\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2 \left(\pi\delta\left(\frac{(\omega_{fi} - \omega)}{2}\right)\right) t$$

Now, using the fact that $\delta(ax) = \frac{1}{|a|}\delta(x)$ and letting $a = 2\hbar$ so that $2\hbar\delta(\hbar(\omega_{fi} \pm \omega)) = \delta\left(\frac{\omega_{fi} \pm \omega}{2}\right)$, the probability becomes

$$P(t) = \frac{W_0^2}{\hbar^2} |\langle\psi_f|\hat{b}|\psi_i\rangle|^2 2\hbar\pi t \delta(\hbar(\omega_{fi} + \omega)) + \frac{W_0^2}{\hbar^2} |\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2 2\hbar\pi t \delta(\hbar(\omega_{fi} - \omega))$$

Since $\hbar\omega_{fi} = (E_f - E_i)$ and the scattering rate is $\frac{1}{\tau} = \frac{dP}{dt}$ we have

$$\frac{1}{\tau} = \frac{2\pi}{\hbar} W_0^2 |\langle\psi_f|\hat{b}|\psi_i\rangle|^2 \delta(E_f - E_i + \hbar\omega) + \frac{2\pi}{\hbar} W_0^2 |\langle\psi_f|\hat{b}^\dagger|\psi_i\rangle|^2 \delta(E_f - E_i - \hbar\omega)$$

in which the first term corresponds to stimulated emission and the second term to absorption of a quanta of energy $\hbar\omega$.